CYCLING WITH NEGATIVE FREQUENCIES

By: Ben, Ooi Zee Min

Two years ago I had the opportunity to visit Singapore's Science Centre. Friends have often related their amazing learning experience there and after visiting, I had to agree, it was remarkable. My theoretical concepts of transmission lines, Lissajous plots, Gauss Law, and many other theories were strengthen by the simple practical applications of these concepts. Having spent the whole day there, I still could not complete my tour of viewing all the exhibits and was a little disappointed. Nevertheless, I would love to visit the centre again for another eye opening adventure.

One of the more interesting applications that caught my attention was the stroboscopic effect on water flowing from a water fountain. Under normal light, this is your normal water fountain. However, when viewed under a strobe light with its frequency tuned to the rate at which the water droplets fall, the water droplets appear to be suspended in mid-air. By adjusting the strobe frequency, one can make the water droplets move slowly up or down. "Negative frequency!," I exclaimed and had to take a video of what I saw. The video can be viewed here. "How is this related to negative frequency?" you may ask. Before revealing my practical viewpoint on this subject, let's explore what is negative frequency.

When dealing with Fourier Transforms of sine or cosine waves, we often come face-to-face with negative frequencies. Remembering Euler's work on sine and cosine waves, we have

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \qquad (1)$$

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \qquad (2)$$

We see that both sine and cosine waves are constructed by the contribution of both positive and negative frequency components and this is true for all real signals. Spectrum analysis on every real signal would indicate that real signals contain both positive and negative frequencies. The complex term $e^{j\omega t}$ is more basic than the sine or cosine waves because $e^{j\omega t}$ consists of only one frequency ω while $sin\omega t$ or $cos\omega t$ really consists of two frequencies ω and $-\omega$. In any case, the negative frequency is required in the construct of any real signals.

Frequency is inversely proportional to the period of a signal. Therefore, intuitively speaking, negative frequencies would imply a negative period which is not practically possible. To be terminologically correct, any signal must be causal. Thus many academics have defined negative frequencies to be a purely mathematical abstraction that does not contain any physical meaning. But in truth, reality follows mathematics faithfully, even though it runs counter to our intuition. To quote from Goethe in "Variational Principles of Mechanics", *was du ererbt von deinen Vaetern hast, erwirb es, um es zu besitzen,* which translates to 'what we inherit from our fathers is ours, but has to be learned again in order to be possessed'. I strongly believe that the works on negative frequency has to have a practical physical meaning.

This brings me to the title of this article, "Cycling with Negative Frequencies". In trying to build upon the practical significance of negative frequencies, I will use the concept of a stroboscope on three exercise bicycles in parallel. Each bicycle has a cyclist; cyclist A is paddling slower than cyclist B, and in turn cyclist B is slower than cyclist C. Just imagine Lance Armstrong on bicycle C while my slightly overweight-self on bicycle A. This is illustrated in Figure 1. If a strobe light is applied to cyclist B, with its strobe frequency tuned to the speed of cyclist B, cyclist B would appear stationary. Using the same strobe frequency on cyclist A, it would appear as though cyclist A is moving backwards and similarly, cyclist C is moving forward. We can now relate this to the frequency spectrum (where cyclist B, appears to be stationary with the strobe light) as the DC component of our spectrum while cyclist A and C are the negative and positive frequencies respectively. What this implies is that negative frequencies do exist! It is just at a different phase rotation than that of positive frequencies. Positive frequencies will have a clockwise (CW) rotation while negative frequencies have a counter clockwise rotation (CCW).

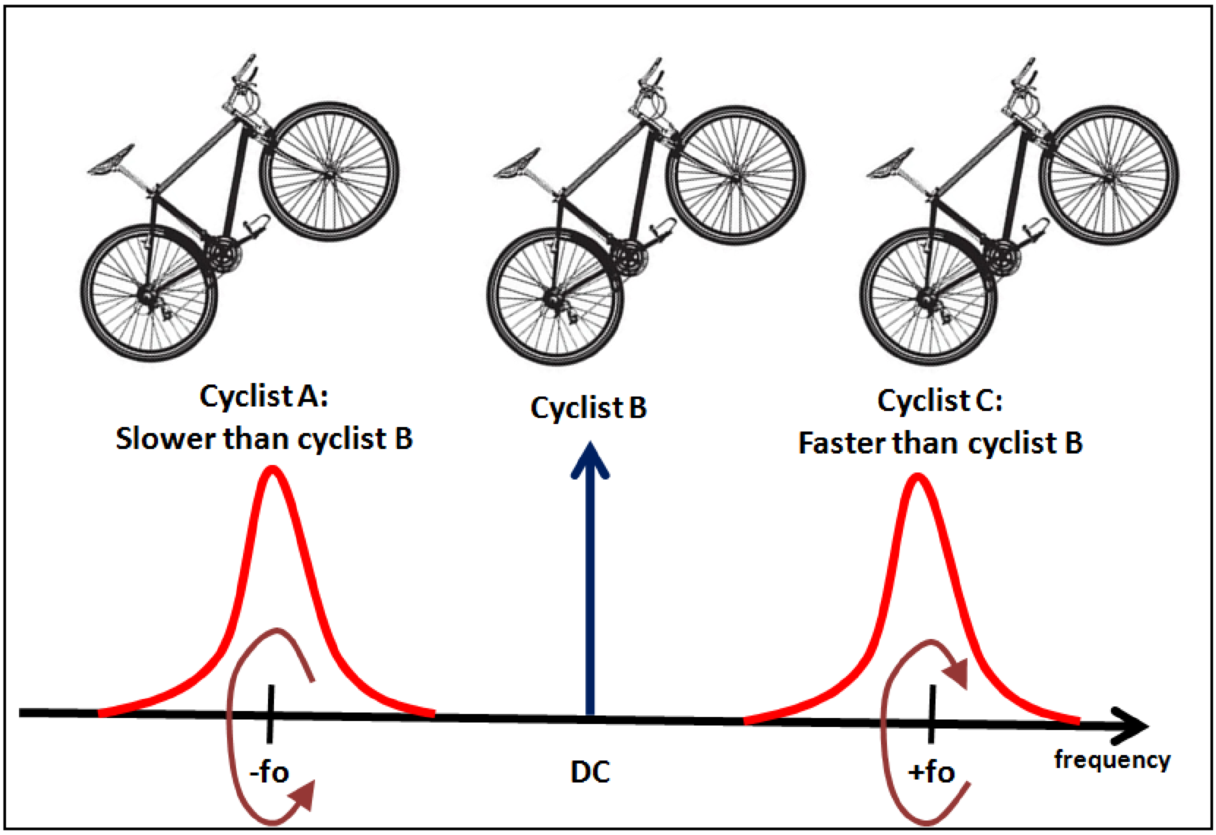

Figure 1: Negative frequency analogy through the use of three bicycles.

Now let's revisit the video. Do you see the negative frequency by now? By adjusting the frequency of the strobe light, the water droplets appear either stationary (DC component), moving forward (+ve frequency), or moving backwards (-ve frequency). Through this example, it is evident that negative frequencies occur with a phase rotation that is opposite to its positive counterpart. The beauty here is that we have avoided any complex mathematical formulation on this topic and yet still have greater insights into this poorly understood concept.

Let's look at Euler's work in equations (1) and (2) (mentioned above) but bearing in mind that negative frequencies are nothing more than a CCW phase rotation. The implication is that both sine and cosine waves (and hence all real sinusoids) consist of a sum of frequencies with equal and opposite circular motion. Plotting the function of both the sine and cosine wave on a 3-dimensional plot would give a clearer picture of what we are trying to understand. Figure 2 is generated in MATLAB with the following codes:

```
t = 0:pi/50:10*pi;
plot3(t,sin(t),cos(t))
xlabel('t')
ylabel('sin(t)')
zlabel('cos(t)')
grid on
axis square
```

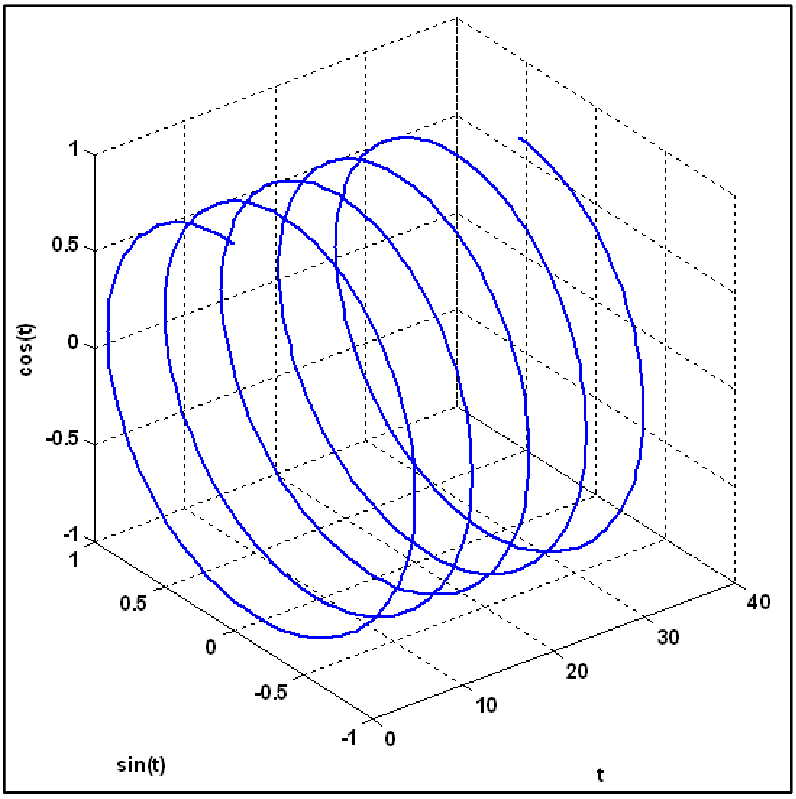

Figure 2: 3-D helical plot of sine and cosine wave against time

This resulting waveform traces an infinite helix (just like a spring). Projecting this waveform from the top is a sine wave and from the side it is a cosine wave as shown in Figure 3.

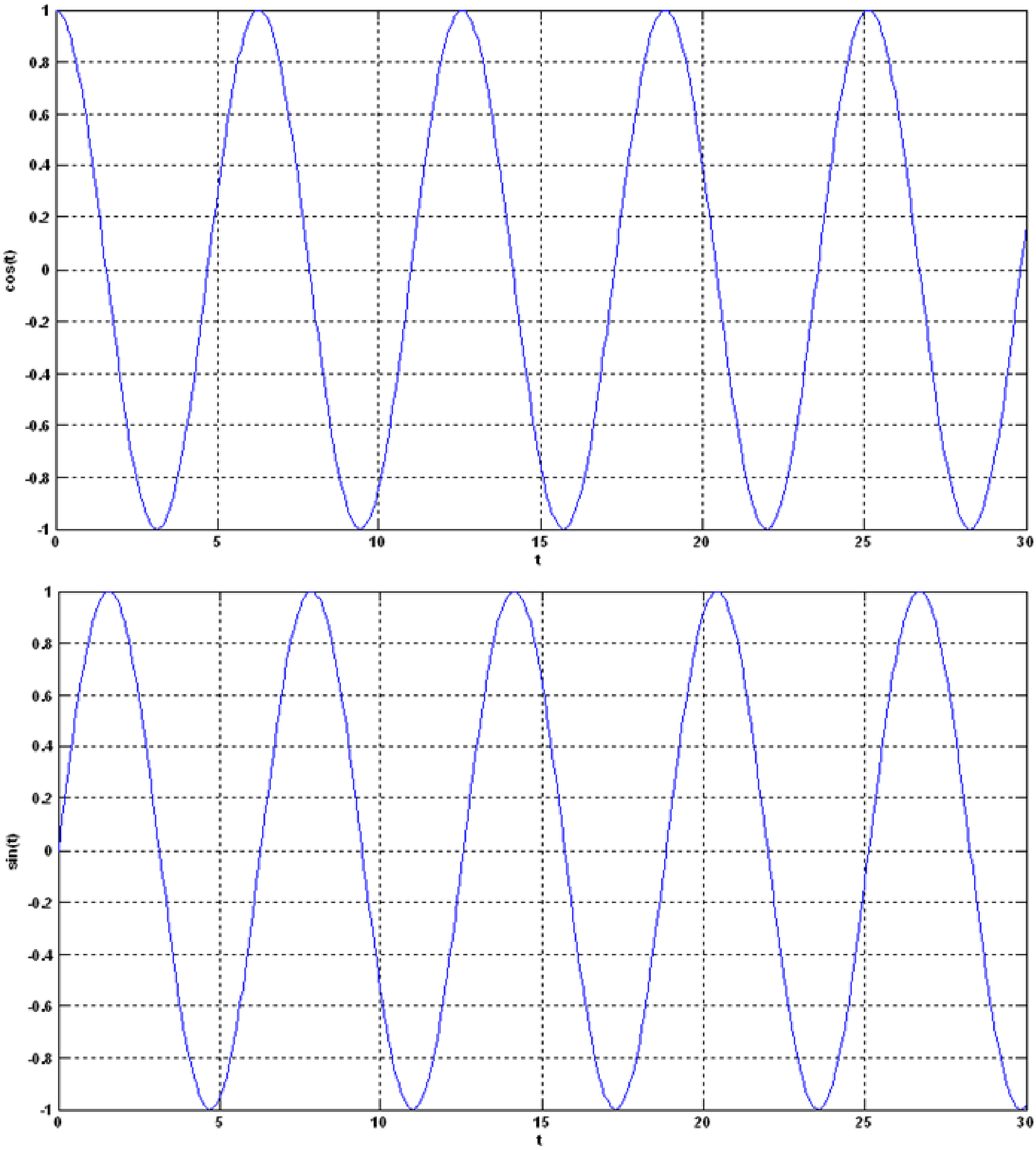

Figure 3: Cosine and sinewave projection of Figure 2.

Things get a little more interesting if viewed from the perpendicular direction to the time axis. The projection simply traces a CW rotation at an angular frequency, ω. This is shown in Figure 4. Negative frequency now is simply the CCW rotation of this projection but the helical structure stays the same. Just because it is negative frequency, we cannot say that it has no physical effect. In fact, the physical effect is quite identical to the positive frequency.

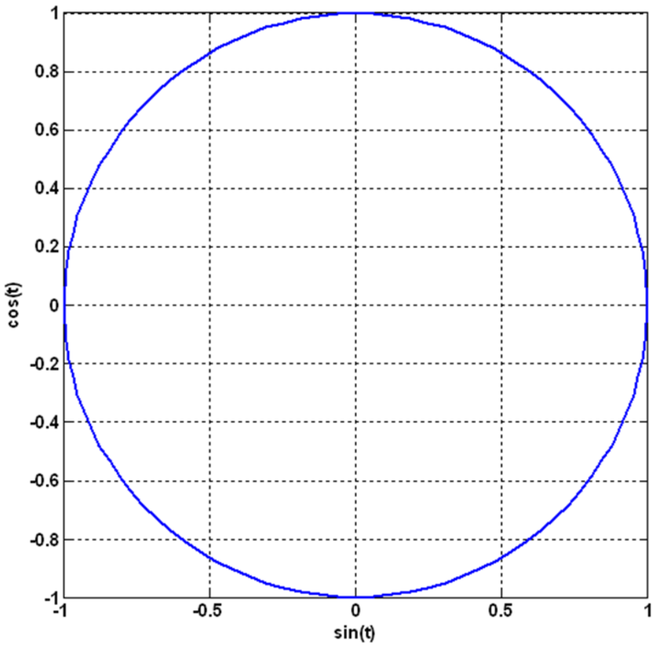

Figure 4: Projection of Figure 2 from the perpendicular direction to the time axis

Negative frequencies are not just mathematical abstractions. They have very real physical meaning when properly understood. Many practical applications deal with negative frequencies. In Doppler radar, any objects moving towards the radar will induce positive frequency, and any objects moving away will induce negative frequency. Direct-conversion receivers have negative image frequencies that must be dealt with carefully. Processing of sound from your microphone will give you both positive and negative frequencies as well. Thus, in many engineering sense, you might encounter negative frequencies.

It is hoped that this short and simple article will help you gain some insights into negative frequencies. If you have any problems with this subject, just go out for a walk, get some fresh air or better still, remember to cycle with negative frequencies.